Incremental construction properties in dimension two—shellability, extendable shellability and vertex decomposability

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Abstract

We give new examples of shellable but not extendably shellable two dimensional simplicial complexes. They include minimal examples, which are smaller than those previously known. We also give examples of shellable but not vertex decomposable two dimensional simplicial complexes. Among them are extendably shellable ones. This shows that neither extendable shellability nor vertex decomposability implies the other. We found these examples by enumerating shellable two dimensional simplicial complexes which are not pseudomanifolds. A rather efficient algorithm for this enumeration is also given.

1 Introduction

A pure simplicial complex is *shellable* if there is a total order of facets according to which the facets can be pasted incrementally in a nice way (see Section 2 for definitions). The notion of shellability was introduced by Bruggesser & Mani [5], who showed the shellability of the boundary complexes of polytopes. Shellability is important both in combinatorial and computational geometry, for example, it was essential for the proof of the upper bound of the number of faces of polytopes [11], or has been used for efficient convex hull construction of polytopes [14]. Shellability has also been studied from algebra through the Stanley-Reisner ring of simplicial complexes [8] [15].

A pure simplicial complex is *extendably shellable* if any sequence of a subset of facets satisfying the condition of being pasted nicely can be continued to a shelling. This means we can make a shelling by pasting facets one by one in a greedy manner. Extendable shellability was defined by Danaraj & Klee [6], who showed that for a 2-pseudomanifold, shellability, extendable shellability, and being a 2-ball or a 2-sphere are equivalent [7]. It is also known that rank 3 (i.e. geometrically, 2-dimensional) matroids are extendably shellable [4]. However, a 3-pseudomanifold, or even the boundary complex of a 4-polytope can be shellable but not extendably shellable [18]. Even in dimension two, if we consider simplicial complexes other than pseudomanifolds, shellable but not extendably shellable examples exist [2, Exercise 7.37] [9, Section 5.3] [15]. Since, for a 1-simplicial complex, or a graph, shellability, extendable shellability and connectivity are equivalent, dimension two is the smallest interesting case to consider. (For more information on shellability, extendable shellability and other combinatorial topological properties, see [3] [6] [16] [17] [18]).

The first topic of this paper is shellable but not extendably shellable 2-simplicial complexes (Section 3). First, we give new examples of such kind. Among them are examples smaller than those in the literature, and we have checked their minimality by enumeration:

Theorem A. *The two* 2*-simplicial complexes* ***V6F9-1**, **2*** *with* 6 *vertices and* 9 *facets are shellable but not extendably shellable* (*Example 2*). *There is no* 2*-simplicial complex with less than* 6 *vertices or less than* 9 *facets having this property.*

Next, we show operations to make larger shellable but not extendably shellable 2-simplicial complexes from smaller ones, and show the relation among the examples with respect to these operations or set inclusion (Propositions 6, 10, Remark 7).

A pure simplicial complex is *vertex decomposable* if there is a total order of vertices according to which the facets including the vertex can be nicely removed. This is another operation for breaking (or constructing) simplicial complexes inductively. Vertex decomposability was first introduced by Billera & Provan [1] [13] in connection with the Hirsch conjecture (see also [3]). Vertex decomposability implies shellability. If all boundary complexes of polytopes were vertex decomposable, then this implied the Hirsch conjecture. However, polyhedra whose boundary complexes are not vertex decomposable (but shellable) have been found [10] [13]. Shellable but not vertex decomposable simplicial complexes begin to exist from 2-simplicial complexes which are not pseudomanifolds [9, Section 5.3] [15].

The second topic of this paper is shellable but not vertex decomposable 2-simplial complexes (Section 4). First, we give new examples of such kind. They have the same size as the smallest example in the literature, and we have checked their minimality by enumeration:

Theorem B. *The three 2-simplicial complexes* `V6F10-1`, `6`, `7` *with* 6 *vertices and* 10 *facets are shellable but not vertex decomposable* (*Example 11*). *There is no 2-simplicial complex with less than* 6 *vertices or with* 6 *vertices and less than* 10 *facets having this property. Furthermore,* `V6F10-1` *is not extendably shellable, but* `V6F10-6`, `7` *are extendably shellable.*

Vertex decomposable but not extendably shellable simplicial complexes have been known (`V6F11-3` [2, Exercise 7.37], for example). However, our extendably shellable but not vertex decomposable examples are new. From these examples, we know that these two properties stronger than shellability do not have logical implications each other:

Corollary C. *Neither extendable shellability nor vertex decomposability implies the other* (*Corollary 12*).

The examples in this paper were generated using a computer. In the final part (Section 5), we propose a rather efficient algorithm to enumerate shellable 2-simplicial complexes which are not pseudomanifolds (Algorithm 16, Theorem 17). It generates one example per each class consisting of those identical with respect to the relabeling of vertices.

The study in this paper is an expansion of [12].

2 Definitions and basic properties

Let $V = \{1, \ldots, n\}$ be a finite set. An (*abstract*) *simplicial complex* is a set Δ consisiting of subsets of V such that if $\sigma \in \Delta$, $\tau \subset \sigma$ then $\tau \in \Delta$. An element of Δ is a *face.* A *facet* is a face maximal with respect to set inclusion. An element of V is a *vertex.* The *dimension* of a face σ is $\dim \sigma = |\sigma| - 1$. The *dimension* of a simplicial complex Δ is $\max_{\sigma \in \Delta} \dim \sigma$. A simplicial complex is *pure* if all facets have the same dimension. A *ridge* of a pure simplicial complex is a face having dimension $\dim \Delta - 1$. A pure simplicial complex is a *pseudomanifold* if any ridge is included in at most two facets. If not, it is a *nonpseudomanifold.* A *boundary ridge* is a ridge contained in only one facet, and a facet containing a boundary ridge is a *boundary facet.* A d-dimensional simplicial complex, pseudomanifold, etc. will be denoted d-simplicial complex, d-pseudomanifold, etc. Two simplicial complexes which become identical by relabeling the vertices are called *isomorphic*, and are regarded as the same.

A *paritial shelling* of a pure d-simplicial complex Δ is a sequence $F_1, \ldots, F_\ell$ of a subset of facets satisfying

$$\overline{F_i} \cap \left(\bigcup_{j=1}^{i-1} \overline{F_j} \right) \quad \text{is a pure } (d-1)\text{-simplicial complex} \quad (1 < i \leq \ell), \tag{$*$}$$

where $\overline{\sigma} = \{\tau \in \Delta : \tau \subset \sigma\}$. A *shelling* is a partial shelling consisting of all of the facets of Δ. A pure simplicial complex is *shellable* if it has a shelling. A partial shelling is *extendable* if there exists a shelling beginning from it. A maximal not extendable partial shelling is called *stuck.* A simplicial complex is *extendably shellable* if any partial shelling is extendable. In other words, extendable shellability means that we can find a shelling by adding facets in a greedy manner.

The *link* of a face $\sigma \in \Delta$ is $\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset\}$. The *deletion* of a face $\sigma \in \Delta$ is $\mathrm{del}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cap \tau = \emptyset\}$. A pure simplicial complex Δ is *vertex decomposable* if it has only one facet, or if it has a vertex i with both $\mathrm{link}_\Delta(\{i\})$ and $\mathrm{del}_\Delta(\{i\})$ vertex decomposable. A vertex decomposable simplicial complex is shellable.

We are interested in the case of dimension two. When a 2-simplicial complex has a 2-dimensional ball (resp. 2-dimensional sphere) as its realization, we simply call it a 2-*ball* (resp. 2-*sphere*). For the top dimensional element

h_3 of the h-vector (or the reduced Euler characteristic), we have

$$h_3 = \#\text{facets} - \#\text{ridges (or edges)} + \#\text{vertices} - 1$$

(see, for example, [17, Chapter 8]).

For a 1-simplicial complex, shellability, extendable shellability, vertex decomposability, and connectivity are equivalent. This kind of simple situation holds until the case of 2-pseudomanifolds:

Theorem 1 **([7]).** *For a 2-pseudomanifold, shellability, extendable shellability, vertex decomposability, and being a 2-ball or a 2-sphere are equivalent.*

3 Shellable but not extendably shellable simplicial complexes

3.1 Examples

We first give shellable but not extendably shellable 2-simplicial complexes found using the enumeration technique in Section 5. They include two known examples. Another larger known example `V7F13` and two smaller examples `V7F12`, `V7F11` made reversing the operation in Proposition 6 are also listed.

Example 2. *The following is a list of shellable but not extendably shellable 2-simplicial complexes. The list covers all such examples with less than 6 vertices, 6 vertices and at most 10 facets, or less than 9 facets. (Such examples do not exist for less than 6 vertices or less than 9 facets.) For the labeling, for example, `V6F9-1` indicates the 1st example with 6 vertices and 9 facets. The 2-simplicial complexes are given as lists of facets, and boundary facets are printed in bold font. After the facets, are given the boundary ridges and stuck partial shellings (unsorted, as sets).*

`V6F9-1`	124, **126**, 134, **135**, 245, 256, 346, 356, 456 *boundary ridges* : 15, 16 *stuck partial shelling* : {124, **126**, 134, **135**}
`V6F9-2`	123, **126**, **135**, 234, 245, 256, 346, 356, 456 *boundary ridges* : 15, 16 *stuck partial shelling* : {123, **126**, **135**, 234}
`V6F10-1` [15]	**123**, 124, **126**, 134, **135**, 245, 256, 346, 356, 456 *boundary ridges* : 15, 16, 23 *stuck partial shelling* : {**123**, 124, **126**, 134, **135**}
`V6F10-2`	124, **126**, 134, **135**, **236**, 245, 256, 346, 356, 456 *boundary ridges* : 15, 16, 23 *stuck partial shelling* : {124, **126**, 134, **135**}
`V6F10-3`	123, **126**, **134**, **135**, 234, 245, 256, 346, 356, 456 *boundary ridges* : 14, 15, 16 *stuck partial shelling* : {123, **126**, **134**, **135**, 234}
`V6F10-4`	123, 126, **135**, **146**, 234, 245, 256, 346, 356, 456 *boundary ridges* : 14, 15, *stuck partial shellings* : {123, 126, **135**, 234}, {123, 126, **135**, **146**}
`V6F10-5`	124, **126**, 134, **135**, **234**, 245, 256, 346, 356, 456 *boundary ridges* : 15, 16, 23 *stuck partial shelling* : {124, **126**, 134, **135**, **234**}
`V6F11-1`	124, **126**, 134, **135**, 235, 236, 245, 256, 346, 356, 456 *boundary ridges* : 15, 16 *stuck partial shelling* : {124, **126**, 134, **135**}
`V6F11-2`	123, 124, **126**, 134, **135**, 234, 245, 256, 346, 356, 456 *boundary ridges* : 15, 16 *stuck partial shelling* : {123, 124, **126**, 134, **135**, 234}
`V6F11-3` [2, Exercise 7.37]	123, 126, 135, 145, 146, 234, 245, 256, 346, 356, 456 *boundary ridges* : $\emptyset$ *stuck partial shelling* : {123, 126, 135, 234}

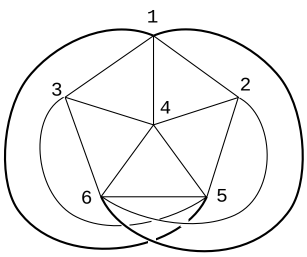
V7F11	$a = 125, b = 126, c = 127, \boldsymbol{e} = \mathbf{145}, f = 167,$ $h = 235, i = 236, \boldsymbol{j} = \mathbf{247}, k = 356, l = 457,$ $m = 567$ *boundary ridges* : $14, 24$ *stuck partial shelling* : $\{\boldsymbol{e}, \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$
V7F12	$a = 125, b = 126, c = 127, \boldsymbol{e} = \mathbf{145}, f = 167,$ $\boldsymbol{g} = \mathbf{234}, h = 235, i = 236, j = 247, k = 356, l = 457,$ $m = 567$ *boundary ridges* : $14, 34$ *stuck partial shellings* : $\{\boldsymbol{e}, \boldsymbol{g}, i, j, l\}, \{\boldsymbol{e}, \boldsymbol{g}, j, l, m\},$ $\{\boldsymbol{e}, j, k, l, m\}$
V7F13 [9, Section 5.3]	$a = 125, b = 126, c = 127, \boldsymbol{d} = \mathbf{134}, e = 145, f = 167,$ $g = 234, h = 235, i = 236, j = 247, k = 356, l = 457,$ $m = 567$ *boundary ridge* : 13 *stuck partial shellings* : $\{a, b, \boldsymbol{d}, e, l\}, \{a, \boldsymbol{d}, e, l, m\},$ $\{e, j, k, l, m\}, \{a, b, c, \boldsymbol{d}, e, f\}, \{\boldsymbol{d}, e, g, i, j, l\},$ $\{\boldsymbol{d}, e, g, j, l, m\}, \{\boldsymbol{d}, g, h, i, j, k\}, \{\boldsymbol{d}, g, h, i, k, m\}$

Checking that these examples are shellable but not extendably shellable was also done using a computer. However, for examples V6F9-1, 2, V6F10-1, 3, 4, 5, V7F11, V7F12 or V7F13, Lemma 3 below gives a proof of not being extendably shellable. This observation can be found, for example, in [16, Section III.2].

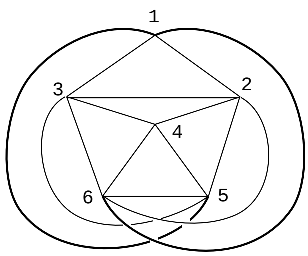
Lemma 3. *Let Δ be a shellable 2-simplicial complex with $h_3 = 0$. Then, the final facet in any of its shelling is a boundary facet. If there exists a proper partial shelling including all of the boundary facets, it does not extend to a shelling of Δ, and Δ is not extendably shellable.*

As can be observed from the examples, stuck partial shellings not including all of the boundary facets also exist.

Remark 4. *Topological drawings of V6F9-1, 2 are shown below. The boundary ridges are drawn in bold lines. The two examples differ only in the way the quadrilateral* 1243 *is triangulated. In V6F9-1 it is triangulated* 124, 134, *whereas in V6F9-2 it is triangulated* 123, 234.



V6F9-1



V6F9-2

Remark 5. *All examples except V6F11-3 in Example 2 can be realized as polyhedral complexes without self intersection in three dimensional space. However, V6F11-3 cannot, because it includes the two dimensional projective space as a subcomplex.*

3.2 Relations

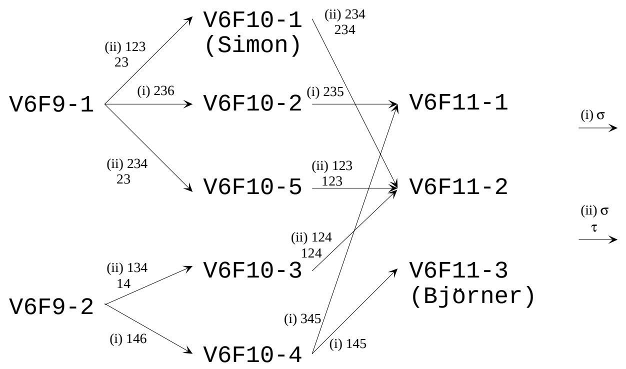
The following proposition gives a way to enlarge a shellable but not extendably shellable example, keeping a stuck partial shelling.

Proposition 6. *Let Δ be a shellable but not extendably shellable pure d-simplicial complex with $F_1, \ldots, F_i$ a stuck partial shelling. Take $\sigma \notin \Delta$, $\dim \sigma = d$ with $\overline{\sigma} \cap \Delta$ being a pure $(d-1)$-simplicial complex.*

(i) *If $\overline{\sigma} \cap \left(\bigcup_{j=1}^{i} \overline{F_j}\right)$ is not a pure $(d-1)$-simplicial complex, then $\Delta \cup \overline{\sigma}$ is shellable but not extendably shellable with $F_1, \ldots, F_i$ a stuck partial shelling.*

(ii) *If* $\overline{\sigma} \cap \left(\bigcup_{j=1}^{i} \overline{F_j}\right)$ *is a pure* $(d-1)$*-simplicial complex and* $\left(\overline{\sigma} \setminus \left(\bigcup_{j=1}^{i} \overline{F_j}\right)\right) \cap \Delta = \emptyset$ *(which is equivalent for the* τ *satisfying* $\left(\overline{\sigma} \setminus \left(\bigcup_{j=1}^{i} \overline{F_j}\right)\right) = \{\eta : \tau \subset \eta \subset \sigma\}$ *being* $\tau \notin \Delta$*), then* $\Delta \cup \overline{\sigma}$ *is shellable but not extendably shellable with* $F_1, \dots, F_i, \sigma$ *a stuck partial shelling.*

Remark 7. *The operations in Proposition 6 defines relations between shellable but not extendably shellable examples. The ten examples* ***V6F9-1****, 2,* ***V6F10-1****, ..., 5,* ***V6F11-1****, ..., 3 are related by these operations as in the figure below.*



Remark 8. *If the vetrices of* σ *belong to* Δ*, and* Δ *includes all of the* d*-subsets (i.e. possible ridges) or one less than that, we do not have to check the condition "*$\overline{\sigma} \cap \Delta$ *being a pure* $(d-1)$*-simplicial complex" in Proposition 6. This is the case for the operations among examples* ***V6F9-1****, 2,* ***V6F10-1****, ..., 5,* ***V6F11-1****, ..., 3 in Remark 7.*

Another relation between the examples to consider is the set inclusion. We show some properties of minimal examples with respect to this relation.

A *homology facet* in a shelling is a facet with any of its proper subface included in some preceding facet in the shelling. If σ is a homology facet in some shelling of a simplicial complex Δ, by simply removing σ, a shelling of $\Delta \setminus \{\sigma\}$ can be made. In a shelling of a 2-simplicial complex, each homology facet contributes one to h_3.

Lemma 9. *Among the shellable but not extendably shellable* 2*-simplicial complexes, let* Δ *be a minimal one with respect to set inclusion. Then any proper partial shelling of* Δ *is extendably shellable. Thus any stuck partial shelling is extendably shellable. Furthermore, stuck partial shellings do not contain* 2*-spheres as subcomplexes.*

Proof. The claims for extendable shellability are clear by the minimality of Δ.

Suppose there was a stuck partial shelling with S the set of its facets including a 2-sphere T as a subcomplex.

Take a shelling of the whole simplicial complex Δ and let σ be the last facet in T. The facet σ is a homology facet of Δ with respect to this shelling. Thus $\Delta \setminus \{\sigma\}$ is shellable.

Next, we show there is a shelling of the stuck partial shelling S with σ a homology facet. Then the simplicial complex with $S \setminus \{\sigma\}$ the facets becomes shellable. Since T is a 2-sphere, by Theorem 1, it has a shelling beginning from σ. By taking the reverse order, we can make a shelling of T ending with σ, and σ becomes a homology facet. Since S is extendably shellable as remarked above, this shelling of T can extend to a shelling of S, and σ is a homology facet also in this extended shelling.

Now, $S \setminus \{\sigma\}$ is a stuck partial shelling in $\Delta \setminus \{\sigma\}$, thus $\Delta \setminus \{\sigma\}$ is not extendably shellable. (A remark redundant to this proof: adding σ to $\Delta \setminus \{\sigma\}$ is a valid operation in Proposition 6.) This contradicts the minimality of Δ. □

Proposition 10. *Among the shellable but not extendably shellable* 2*-simplicial complexes, let* Δ *be a minimal one with respect to set inclusion. Then* Δ *does not contain* 2*-spheres as subcomplexes.*

Proof. Suppose Δ included a 2-sphere T as a subcomplex. Since stuck partial shellings do not contain 2-spheres (Lemma 9), T is not included in any of the stuck partial shellings. Hence a shelling of T can be extended to a shelling of the whole simplicial complex Δ.

Take a stuck partial shelling S of Δ. Since S does not contain a 2-sphere, there should exist a 2-simplex $\sigma \in T \setminus S$.

Similar as in the proof of Lemma 9, T has a shelling ending with σ, with σ a homology facet. As remarked above, we can extend this shelling of T to a shelling of the whole simplicial complex Δ, and σ is still a homology facet. Thus $\Delta \setminus \{\sigma\}$ is shellable. The stuck partial shelling S of Δ is a stuck partial shelling of $\Delta \setminus \{\sigma\}$, thus $\Delta \setminus \{\sigma\}$ is not extendably shellable. (A remark redundant to this proof: adding σ to $\Delta \setminus \{\sigma\}$ is a valid operation in Proposition 6.) This contradicts the minimality of Δ. □

Remark that `V6F9-1`, `2`, `V7F11` are minimal with respect to set inclusion. Other interesting questions to consider might be (1) if minimal examples have $h_3 = 0$ (i.e. do not contain "homology 2-spheres"), (2) if minimal examples have the least number of facets for fixed number of vertices, or (3) if stuck partial shellings of such examples contain all of the boundary facets. Dealing with the relations by the operations in Proposition 6 is another interseting subject.

4 Shellable but not vertex decomposable simplicial complexes

We first give shellable but not vertex decomposable 2-simplicial complexes found using the enumeration technique in Section 5. They include one known example. Another larger known example `V7F13` is also listed.

Example 11. *The following is a list of shellable but not vertex decomposable* 2*-simplicial complexes. The list covers all such examples with less than* 6 *vertices,* 6 *vertices and at most* 10 *facets, or less than* 9 *facets.* (*Such examples do not exist for less than* 6 *vertices or less than* 9 *facets.*) *The* 2*-simplicial complexes are given as lists of facets, and boundary facets are printed in bold font. After the facets, are given the boundary ridges. Examples* ***V6F10-6***, *7 are not vertex decomposable but extendably shellable.*

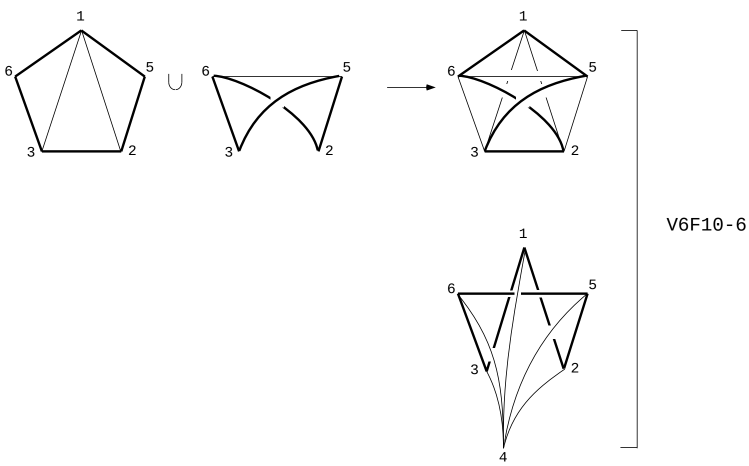
`V6F10-1` [15]	*see Example 2*
`V6F10-6`	$\mathbf{123}, 124, \mathbf{125}, 134, \mathbf{136}, 245, \mathbf{256}, 346, \mathbf{356}, 456$ *boundary ridges* : $15, 16, 23, 26, 35$
`V6F10-7`	$123, 125, \mathbf{126}, 134, 145, \mathbf{234}, 256, 346, \mathbf{356}, 456$ *boundary ridges* : $16, 24, 35$
`V7F13` [9, Section 5.3]	*see Example 2*

Checking that these examples are shellable but not vertex decomposable was also done using a computer.

Extendable shellability and vertex decomposability are both properties stronger than shellability. Vertex decomposable but not extendably shellable simplicial complexes have been known (`V6F11-3` [2, Exercise 7.37], for example). On the other hand, examples `V6F10-6`, `7` show the existence of extendably shellable but not vertex decomposable ones. Thus we know there are no implication between these two properties.

Corollary 12. *Neither extendable shellability nor vertex decomposability implies the other.*

Remark 13. *A topological drawing of* ***V6F10-6*** *is shown below. Boundary ridges are drawn in bold lines.*



Remark 14. *All examples in Example 11 can be realized as polyhedral complexes without self intersection in three dimensional space.*

5 Enumeration of shellable nonpseudomanifolds

We call the 2-simplicial complex with five vertices $1, \ldots, 5$ and three facets 123, 124, 125 the *initial simplicial complex*, and denote it by Δ_{initial}. This is the minimal 2-nonpseudomanifold. For a 2-simplicial complex, we also call a subcomplex isomorphic to Δ_{initial} an *initial simplicial complex.*

Proposition 15. *For any shellable* 2*-nonpseudomanifold* Δ*, there exists a shelling beginning from one of its initial simplicial complexes.*

Proof. Omitted for this version. □

Algorithm 16. *Begin from* Δ_{initial}*. Add facets one by one in a shelling manner (i.e. satisfying* $(*)$ *in Section 2). The shellable nonpseudomanifolds with* v *vertices and* f *facets are made from those with* v *or* $v-1$ *vertices and* $f-1$ *facets.*

Theorem 17. *Algorithm 16 enumerates shellable* 2*-simplicial complexes which are not pseudomanifolds.*

Proof. Proposition 15. □

During the enumeration, for each size of vertices and facets, we only want to output one simplicial complex among the isomorphic ones. This can be done using the following lemma.

Lemma 18. *We can find a canonical labeling with respect to isomorphism of a* 2*-simplicial complex with* v *vertices and* f *facets in* $O(v!vf)$ *time.*

Proof. Consider the vertex facet incidence matrix. Make all copies for the $v!$ different vertex labelings. Remark that $v < f$ for the examples we are interested in. Use radix sort. □

Finally, the numbers of isomorphism classes of shellable nonpseudomanifolds we enumerated are shown in Table 3.

# of vertices	# of facets																	
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5	1	1	3	4	4	2	1	1										
6		2	8	23	51	100	170	254	269	233	157	93	43	21	7	3	1	1
7			8	42	167	535	1628	···										
8				27	217	1114	···											
9					109	1106	···											
10						447	···											

Table 3: The number of isomorphism classes of shellable two dimensional nonpseudomanifolds with specified numbers of vertices and facets.

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