Incremental construction properties in dimension two— shellability, extendable shellability and vertex decomposability ⋆

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Abstract

We give new examples of shellable, but not extendably shellable two dimensional simplicial complexes. They include minimal examples that are smaller than those previously known. We also give new examples of shellable, but not vertex decomposable two dimensional simplicial complexes, including extendably shellable ones. This shows that neither extendable shellability nor vertex decomposability implies the other. We found these examples by enumerating shellable two dimensional simplicial complexes that are not pseudomanifolds.

Key words: incremental construction, shellability, extendable shellability, vertex decomposability, combinatorial topology, enumeration

A pure n-dimensional simplicial complex $\mathcal{X}$—or, for short, an n-complex—is *shellable* if its facets (i.e., n-simplices) can be ordered linearly so that the intersection of any facet with the union of its preceding facets (relative to that order) is a union of $(n-1)$-simplices, and any such linear order is called a shelling of $\mathcal{X}$ (see [1,4] or [5] for details and examples). An n-complex $\mathcal{X}$ is *extendably shellable* if any shelling of an n-subcomplex of $\mathcal{X}$ can be continued to a shelling of $\mathcal{X}$.

Consider the following five 2-complexes defined in terms of a complete list of their facets (see [5] for further examples). In the table, 1-simplices that are contained in one facet, only, are indicated by bold letters. For shellable, but

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not extendably shellable complexes, the stuck partial shellings (unsorted, as sets, which happen to be unique in these examples) are also provided:

`V6F9-1`	124, **126**, 134, **135**, 245, 256, 346, 356, 456 stuck partial shelling : {124, **126**, 134, **135**}
`V6F9-2`	123, **126**, **135**, 234, 245, 256, 346, 356, 456 stuck partial shelling : {123, **126**, **135**, 234}
`V6F10-1` [3]	**123**, 124, **126**, 134, **135**, 245, 256, 346, 356, 456 stuck partial shelling : {**123**, 124, **126**, 134, **135**}
`V6F10-6`	**123**, 124, **125**, 134, **136**, 245, **256**, 346, **356**, 456
`V6F10-7`	123, 125, **126**, 134, 145, **234**, 256, 346, **356**, 456

Theorem A *The two complexes* `V6F9-1`, `2` *having* 6 *vertices and* 9 *facets are shellable, but not extendably shellable. There is no* 2*-complex with less than* 6 *vertices or less than* 9 *facets having this property.*

An n-complex $\mathcal{X}$ is *vertex decomposable* if it has only one facet, or if it has a vertex v such that both, its *link* $\text{link}_{\mathcal{X}}(v) := \{X - \{v\} \mid v \in X \in \mathcal{X}\}$ as well as its *deletion* $\text{del}_{\mathcal{X}}(v) := \{X \in \mathcal{X} \mid v \notin X\}$, are vertex decomposable. Vertex decomposability is known to imply shellability.

Theorem B *The three complexes* `V6F10-1`, `6`, `7` *having* 6 *vertices and* 10 *facets are shellable, but not vertex decomposable. There is no* 2*-complex with less than* 6 *vertices or with* 6 *vertices and less than* 10 *facets having this property. Furthermore,* `V6F10-1` *is not extendably shellable while* `V6F10-6`, `7` *are extendably shellable.*

Vertex decomposable, yet not extendably shellable complexes have been known while our examples of extendably shellable, yet not vertex decomposable complexes appear to be new. Together, these examples demonstrate that none of these two stronger properties implies other.

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