Nonregular triangulations, view graphs of triangulations, and linear programming duality

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Abstract. For a triangulation and a point, we define a directed graph representing the order of the maximal dimensional simplices in the triangulation viewed from the point. We prove that triangulations having a cycle the reverse of which is not a cycle in this graph viewed from some point are forming a (proper) subclass of nonregular triangulations. We use linear programming duality to investigate further properties of nonregular triangulations in connection with this graph.

1 Introduction

Let $\mathcal{A} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} \subset \mathbb{R}^d$ be a point configuration with its convex hull $\mathrm{conv}(\mathcal{A})$ being a d-dimensional polytope. A *triangulation* Δ *of* $\mathcal{A}$ is a geometric simplicial complex with its vertices among $\mathcal{A}$ and the union of its faces equal to $\mathrm{conv}(\mathcal{A})$. A triangulation is *regular* (or *coherent*) if it can appear as the projection of the lower faces of the boundary complex of a $(d+1)$-dimensional polytope in $\mathbb{R}^{d+1}$. If not, the triangulation is *nonregular.* (See, for example, [6] [11].)

Starting from the study of generalized hypergeometric functions, Gel'fand, Kapranov & Zelevinskiĭ showed that regular triangulations of a point configuration are forming a polytopal structure described by the secondary polytope [4] [5]. In connection with Gröbner bases, Sturmfels showed that initial ideals for the affine toric ideal determined by a point configuration correspond to the regular triangulations of the point configuration [9] [10]. Regular triangulations are a generalization of the Delaunay triangulation well known in computational geometry, and have also been used extensively in this field [2].

Though nonregular triangulations are known to be behaving differently from regular triangulations, they are not well understood yet. Santos showed a nonregular triangulation with no flips indicating that a flip graph can be disconnected, which never happens when restricted to regular triangulations [8]. Ohsugi & Hibi showed the existence of a point configuration with no unimodular regular triangulations, but with a unimodular nonregular triangulation [7]. Also, de Loera, Hoşten, Santos & Sturmfels showed that cyclic polytopes can have exponential number of nonregular triangulations compared to polynomial number of regular ones [1]. The aim of this paper is to put some insight into nonregular triangulations.

In the sequel, we fix a triangulation Δ. For the triangulation Δ and a point $\boldsymbol{v}$ in $\mathbb{R}^d$, we define the *graph* $G_{\boldsymbol{v}}$ *of* Δ *viewed from* $\boldsymbol{v}$ as the directed graph with its vertices corresponding to the d-simplices in Δ and a directed edge $\overrightarrow{\sigma\tau}$ existing when σ, τ are adjacent and $\boldsymbol{v}$ belongs to the closed halfspace having the affine hull $\operatorname{aff}(\sigma \cap \tau)$ as its boundary and including σ. When $\boldsymbol{v} \in \operatorname{aff}(\sigma \cap \tau)$, both edges $\overrightarrow{\sigma\tau}, \overrightarrow{\tau\sigma}$ appear in $G_{\boldsymbol{v}}$. The graph $G_{\boldsymbol{v}}$ is a directed graph whose underlying undirected graph is the adjacency graph of the d-simplices in Δ. Of course, $G_{\boldsymbol{v}}$ might differ for different choices of $\boldsymbol{v}$. Though there are infinitely many choices of viewpoints $\boldsymbol{v}$, there are only finitely many view graphs $G_{\boldsymbol{v}}$.

A sequence of vertices $\sigma_1, \sigma_2, \dots, \sigma_i, \sigma_1$ in $G_{\boldsymbol{v}}$ forms a *cycle* when $\overrightarrow{\sigma_1\sigma_2}, \dots, \overrightarrow{\sigma_{i-1}\sigma_i}, \overrightarrow{\sigma_i\sigma_1}$ are edges of $G_{\boldsymbol{v}}$ and $\sigma_i \neq \sigma_j$ for $i \neq j$. We define a cycle $\sigma_1, \sigma_2, \dots, \sigma_i, \sigma_1$ to be *contradicting* when the reverse sequence $\sigma_1, \sigma_i, \dots, \sigma_2, \sigma_1$ is not a cycle in $G_{\boldsymbol{v}}$. For vertices $\sigma_1, \dots, \sigma_i$ in $G_{\boldsymbol{v}}$, the edges $\overrightarrow{\sigma_1\sigma_2}, \dots, \overrightarrow{\sigma_{i-1}\sigma_i}, \overrightarrow{\sigma_2\sigma_1}, \dots, \overrightarrow{\sigma_i\sigma_{i-1}}$ exist if and only if $\boldsymbol{v} \in \operatorname{aff}(\sigma_1 \cap \dots \cap \sigma_i)$.

The regularity of a triangulation can be stated as a linear programming problem, so regularity and linear programming obviously have a connection. An interesting point in our argument is that we use linear programming duality to analyze in further detail some properties of nonregular triangulations.

For any triangulation, the condition of regularity can be written as a linear programming problem as follows. Let the variable $\boldsymbol{w} = (w_1, \dots, w_n)$ represent the $(d+1)$-coordinates of the lifting (or weight) of the vertices $\boldsymbol{p}_1, \dots, \boldsymbol{p}_n$, such that the triangulation is lifted to a piecewise linear function $f_{\boldsymbol{w}}$ from $\operatorname{conv}(\mathcal{A})$ to $\mathbb{R}$. For each $(d-1)$-simplex in Δ not in the boundary of the convex hull $\operatorname{conv}(\mathcal{A})$, the local convexity of $f_{\boldsymbol{w}}$ can be expressed by a linear inequality involving only the vertices of the two adjacent d-simplices in Δ (see Section 2.1). Gather the inequality constraints that correspond to each such interior $(d-1)$-simplex. Altogether, we get a system of inequalities $A\boldsymbol{w} > \boldsymbol{0}$ ($\boldsymbol{0}$ is the zero vector), and the triangulation is regular if and only if this has a solution. Easily, this is equivalent to $A\boldsymbol{w} \geq \boldsymbol{1}$ ($\boldsymbol{1}$ is the vector with all entries one) having a solution. Thus, by linear programming duality (or Farkas' lemma), the triangulation is nonregular if and only if the *dual problem* $\boldsymbol{u}A = \boldsymbol{0}$, $\boldsymbol{u} \geq \boldsymbol{0}$ has a nonzero solution.

Our main theorem constructs a nonzero solution of the dual problem combinatorially and explicitly from a contradicting cycle in a graph of the triangulation viewed from some point.

Theorem. *For a triangulation Δ, if a graph $G_{\boldsymbol{v}}$ viewed from some point $\boldsymbol{v}$ contains a contradicting cycle, in correspondence with this cycle, we can make a nonzero solution of the dual problem stated above. Thus, Δ is nonregular. The support set* (*i.e. collection of nonzero elements*) *of this solution becomes a subset of the edges forming the cycle. On the other hand, the reverse of this claim is not true. There exists a nonregular triangulation with none of its view graphs $G_{\boldsymbol{v}}$ containing a contradicting cycle.* (*See Example* 5)

The theorem says that triangulations containing a contradicting cycle in its graph $G_{\boldsymbol{v}}$ viewed from some point $\boldsymbol{v}$ are forming a (proper) subclass of nonregular triangulations. This subclass of triangulations is interesting in that their nonregularity are described more combinatorially using graphs. On the other hand, regularity or nonregularity, defined by linear inequalities, are of continuous nature. This is the first attempt to give a (combinatorial) subclass of nonregular triangulations. Even if we consider contradicting closed paths instead of contradicting cycles, allowing to pass the same vertex more than once, the class of the triangulations having such contradicting thing in its view graph does not change, because any contradicting closed path includes a contradicting cycle.

Checking that Example 5 is a counterexample for the reverse of the implication in the theorem (i.e. the view graph from any viewpoint does not contain a contradicting cycle), can be done by extensive enumeration of view graphs. However, by describing nonregularity as a linear programming problem, and using linear programming duality, we prove the counterexample in a more elegant way.

A similar but different directed graph of a triangulation viewed from a point has been studied by Edelsbrunner [3]. This was in the context of computer vision, and his graph represents the in_front/behind relation among simplices of any dimension, even not adjacent to each other. When our graph and the restriction of Edelsbrunner's graph to d-simplices are compared, neither includes the other in general. However, if we take the transitive closure of our graph, it includes his graph as a (possibly proper) subgraph. Our graph might be more appropriate in describing combinatorial structures of triangulations, because their underlying undirected graphs are the adjacency graphs of d-simplices. Edelsbrunner proves that if a triangulation is regular, his graph viewed from any point is "acyclic". The line shelling argument in a note there gives a proof for the contrapositive of our theorem, but without explicit construction of a solution of the dual problem.

We first prepare basic results, and then prove our main theorem (Section 2). Finally, we give illustrative examples and a counterexample for the reverse of the main theorem (Section 3).

2 Regularity, linear programming, and duality

2.1 Inequalities for regularity

A triangulation Δ of the point configuration $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ is regular if there exists a lifting (or weight) $w_1, \ldots, w_n \in \mathbb{R}$ such that the projection with respect to the x_{d+1} axis of the lower faces of the boundary complex of the $(d+1)$-dimensional polytope $\mathrm{conv}(\binom{\boldsymbol{p}_1}{w_1}, \ldots, \binom{\boldsymbol{p}_n}{w_n})$ becomes Δ. This condition on the lifting is equivalent to the condition that the function from $\mathrm{conv}(\mathcal{A})$ to $\mathbb{R}$ obtained by interpolating the lifting according to the triangulation in a piecewise linear fashion is convex. This implies (in fact, is equivalent to) the local convexity of this function in the neighborhood of every $(d-1)$-simplex in Δ which is not on the boundary of Δ. These conditions can be described by inequalities with $w_1, \ldots, w_n$ the variables.

A global criterion for convexity is therefore as follows:

– For each d-simplex $\mathrm{conv}(\boldsymbol{p}_{i_0},\ldots,\boldsymbol{p}_{i_d})$ in Δ, and any point $\boldsymbol{p}_j \notin \{\boldsymbol{p}_{i_0},\ldots,\boldsymbol{p}_{i_d}\}$, the lifted point $\binom{\boldsymbol{p}_j}{w_j}$ is above the hyperplane $\mathrm{aff}(\binom{\boldsymbol{p}_{i_0}}{w_{i_0}},\ldots,\binom{\boldsymbol{p}_{i_d}}{w_{i_d}})$ in $\mathbb{R}^{d+1}$:

$$\begin{vmatrix} 1 & \cdots & 1 \\ \boldsymbol{p}_{i_0} & \cdots & \boldsymbol{p}_{i_d} \end{vmatrix} \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \boldsymbol{p}_{i_0} & \cdots & \boldsymbol{p}_{i_d} & \boldsymbol{p}_j \\ w_{i_0} & \cdots & w_{i_d} & w_j \end{vmatrix} > 0.$$

A local criterion for convexity can be expressed with much fewer inequalities as follows:

– For each interior $(d-1)$-simplex $\mathrm{conv}(\boldsymbol{p}_{i_1},\ldots,\boldsymbol{p}_{i_d})$ in Δ, where the two d-simplices $\mathrm{conv}(\boldsymbol{p}_{i_0},\boldsymbol{p}_{i_1},\ldots,\boldsymbol{p}_{i_d})$ and $\mathrm{conv}(\boldsymbol{p}_{i_1},\ldots,\boldsymbol{p}_{i_d},\boldsymbol{p}_{i_{d+1}})$ are intersecting, the lifted point $\binom{\boldsymbol{p}_{i_{d+1}}}{w_{i_{d+1}}}$ is above the hyperplane $\mathrm{aff}(\binom{\boldsymbol{p}_{i_0}}{w_{i_0}},\ldots,\binom{\boldsymbol{p}_{i_d}}{w_{i_d}})$ in $\mathbb{R}^{d+1}$:

$$\begin{vmatrix} 1 & \cdots & 1 \\ \boldsymbol{p}_{i_0} & \cdots & \boldsymbol{p}_{i_d} \end{vmatrix} \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \boldsymbol{p}_{i_0} & \cdots & \boldsymbol{p}_{i_d} & \boldsymbol{p}_{i_{d+1}} \\ w_{i_0} & \cdots & w_{i_d} & w_{i_{d+1}} \end{vmatrix} > 0. \qquad (*)$$

The equivalence of these two convexity conditions is proved easily by reducing to the one dimensional case.

The collection of inequalities $(*)$ for all interior $(d-1)$-simplices in Δ form a linear program which we denote by

$$A\boldsymbol{w} > \boldsymbol{0}.$$

We say the matrix A of this linear program *represents the regularity of* Δ. Note that this program has solutions if and only if the program $A\boldsymbol{w} \geq \boldsymbol{1}$ has solutions. Let m be the number of interior $(d-1)$-simplices in Δ. The matrix A is an $m \times n$ matrix.

Lemma 1 *For a triangulation Δ, and the matrix A representing its regularity, Δ is regular if and only if there exists $\boldsymbol{w} \in \mathbb{R}^n$ such that $A\boldsymbol{w} \geq \boldsymbol{1}$. By linear programming duality (or Farkas' lemma), Δ is nonregular if and only if there exists $\boldsymbol{u} \geq \boldsymbol{0}$, $\boldsymbol{u} \neq \boldsymbol{0}$ such that $\boldsymbol{u}A = \boldsymbol{0}$.*

Thus, the (non)regularity of Δ can be judged by the existence of a nonzero point in the polyhedron $\{\boldsymbol{u} \geq \boldsymbol{0} : \boldsymbol{u}A = \boldsymbol{0}\} \subset \mathbb{R}^m$ of the set of solutions of the dual problem.

2.2 A nonzero solution of the dual problem from a contradicting cycle

Here, we give an explicit construction of a nonzero solution of the dual problem $\boldsymbol{u}A = \boldsymbol{0}, \boldsymbol{u} \geq \boldsymbol{0}$, from a contradicting cycle in the graph $G_{\boldsymbol{v}}$ viewed from some point $\boldsymbol{v}$. For $\boldsymbol{v} \in \mathbb{R}^d$, a d-simplex σ in Δ, and $\boldsymbol{w} \in \mathbb{R}^n$, define $x_{d+1}(\boldsymbol{v},\sigma,\boldsymbol{w})$ as follows: consider the projection along the $(d+1)$-coordinate of the point $\boldsymbol{v}$ to the affine hull of $f_{\boldsymbol{w}}(\sigma)$, the lifting of the d-simplex σ by $\boldsymbol{w}$, in $\mathbb{R}^{d+1}$, and let $x_{d+1}(\boldsymbol{v},\sigma,\boldsymbol{w})$ be the x_{d+1} coordinate of this point.

Lemma 2 *Let Δ be a triangulation, A the matrix representing its regularity, and $\boldsymbol{v} \in \mathbb{R}^d$. For an edge $\overrightarrow{\sigma\tau}$ in the graph $G_{\boldsymbol{v}}$ viewed from $\boldsymbol{v}$, there exists a constant $\alpha_{\sigma\cap\tau} \geq 0$ such that*

$$x_{d+1}(\boldsymbol{v}, \sigma, \boldsymbol{w}) - x_{d+1}(\boldsymbol{v}, \tau, \boldsymbol{w}) = \alpha_{\sigma\cap\tau} A_{\sigma\cap\tau,*} \boldsymbol{w} \qquad (\textit{for any } \boldsymbol{w} \in \mathbb{R}^n),$$

where $A_{\sigma\cap\tau,*}$ is the row of A corresponding to the interior $(d-1)$-simplex $\sigma \cap \tau$ in Δ. Furthermore, $\boldsymbol{v} \in \mathrm{aff}(\sigma \cap \tau)$ if and only if $\alpha_{\sigma\cap\tau} = 0$.

Proof. Straightforward.

We now construct a nonzero solution of the dual problem from a contradicting cycle. This will prove the forward implication in our main theorem.

Proof. **(main theorem)** Suppose we have a contradicting cycle $\sigma_1, \sigma_2, \ldots, \sigma_i, \sigma_1$ in $G_{\boldsymbol{v}}$. By Lemma 2, we can find $\alpha_{\sigma_1\cap\sigma_2}, \ldots, \alpha_{\sigma_i\cap\sigma_1} \geq 0$, or their collection as a vector $\boldsymbol{\alpha} \geq \mathbf{0}$, satisfying for any $\boldsymbol{w} \in \mathbb{R}^n$,

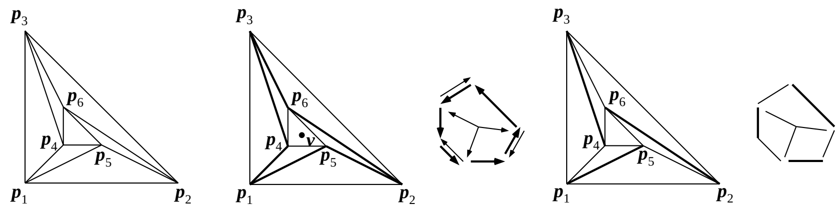
$$\begin{aligned}
& x_{d+1}(\boldsymbol{v}, \sigma_1, \boldsymbol{w}) - x_{d+1}(\boldsymbol{v}, \sigma_2, \boldsymbol{w}) \\
& \quad \cdots \\
& \quad + x_{d+1}(\boldsymbol{v}, \sigma_i, \boldsymbol{w}) - x_{d+1}(\boldsymbol{v}, \sigma_1, \boldsymbol{w}) \\
&= \alpha_{\sigma_1\cap\sigma_2} A_{\sigma_1\cap\sigma_2,*} \boldsymbol{w} \\
& \quad \cdots \\
& \quad + \alpha_{\sigma_i\cap\sigma_1} A_{\sigma_i\cap\sigma_1,*} \boldsymbol{w} \\
&= \boldsymbol{\alpha} A \boldsymbol{w} \\
&= 0.
\end{aligned}$$

(The elements corresponding to the edges not in the cycle have 0 as their value in vector $\boldsymbol{\alpha}$.) Thus, $\boldsymbol{\alpha} A = \mathbf{0}$. Since we took a contradicting cycle, by Lemma 2, $\boldsymbol{\alpha} \neq \mathbf{0}$. Hence, we obtain a nonzero solution of the dual problem $\boldsymbol{u}A = \mathbf{0}, \boldsymbol{u} \geq \mathbf{0}$. This together with Lemma 1 proves the forward implication in the main theorem.

2.3 Recognizing nonregularity or finding contradicting cycles

Judging whether the given triangulation Δ is (non)regular reduces to judging whether the system of inequalities $A\boldsymbol{w} \geq \mathbf{1}$ has a solution $\boldsymbol{w}$, where the matrix A represents the regularity of Δ in the sense described above. This is a linear programming problem, and can be computed in polynomial time for fixed dimension d, for example, using interior point method.

One way to judge if a triangulation Δ has a contradicting cycle in some view graph $G_{\boldsymbol{v}}$ is to enumerate all possible view graphs and enumerate the cycles there. The generation of view graphs can be done, for example, by generating all graphs viewed from the minimal cells in the hyperplane arrangement made by the affine hulls of the interior $(d-1)$-simplices in Δ.



(a) triangulation (b) graph viewed from $\boldsymbol{v}$ (c) support of the dual solutions

Fig. 1. Example 3.

3 Examples

Example 3 (A nonregular triangulation with 6 vertices) For the point configuration

$$\begin{array}{lll} \boldsymbol{p}_1 = (0\,0), & \boldsymbol{p}_2 = (4\,0), & \boldsymbol{p}_3 = (0\,4), \\ \boldsymbol{p}_4 = (1\,1), & \boldsymbol{p}_5 = (2\,1), & \boldsymbol{p}_6 = (1\,2), \end{array}$$

we consider the triangulation Δ indicated in Fig. 1(a) below. The graph $G_{\boldsymbol{v}}$ viewed from $\boldsymbol{v} = (\frac{4}{3}\,\frac{4}{3})$ is in Fig. 1(b), since $\boldsymbol{v}$ lies on $\boldsymbol{p}_1\boldsymbol{p}_4$, $\boldsymbol{p}_2\boldsymbol{p}_5$, and $\boldsymbol{p}_3\boldsymbol{p}_6$. It has one contradicting cycle $\boldsymbol{p}_1\boldsymbol{p}_4\boldsymbol{p}_5, \boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_5, \boldsymbol{p}_2\boldsymbol{p}_5\boldsymbol{p}_6, \boldsymbol{p}_2\boldsymbol{p}_3\boldsymbol{p}_6, \boldsymbol{p}_3\boldsymbol{p}_4\boldsymbol{p}_6, \boldsymbol{p}_1\boldsymbol{p}_3\boldsymbol{p}_4, \boldsymbol{p}_1\boldsymbol{p}_4\boldsymbol{p}_5$ denoted by bold edges. The matrix representing the regularity of Δ is

$$A = \begin{array}{c|cccccc} & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ \hline \boldsymbol{p}_1\boldsymbol{p}_4 & 3 & & 1 & -8 & 4 & \\ \boldsymbol{p}_1\boldsymbol{p}_5 & -1 & 1 & & 4 & -4 & \\ \boldsymbol{p}_2\boldsymbol{p}_5 & 1 & 3 & & & -8 & 4 \\ \boldsymbol{p}_2\boldsymbol{p}_6 & & -1 & 1 & & 4 & -4 \\ \boldsymbol{p}_3\boldsymbol{p}_4 & 1 & & -1 & -4 & & 4 \\ \boldsymbol{p}_3\boldsymbol{p}_6 & & 1 & 3 & 4 & & -8 \\ \boldsymbol{p}_4\boldsymbol{p}_5 & 1 & & & -3 & 1 & 1 \\ \boldsymbol{p}_4\boldsymbol{p}_6 & & & 1 & 1 & 1 & -3 \\ \boldsymbol{p}_5\boldsymbol{p}_6 & & 1 & & 1 & -3 & 1 \end{array}.$$

The polyhedron of the solutions of the dual problem is a single ray

$$\{\boldsymbol{u} \geq \boldsymbol{0} : A\boldsymbol{u} = \boldsymbol{0}\} = \mathbb{R}_{\geq 0}(0\,1\,0\,1\,1\,0\,0\,0\,0),$$

where interior 1-simplices are indexed lexicographically. The support of the nonzero solutions is denoted by bold edges in Fig. 1(c). Remark that they are included in the (underlying undirected) edges of the contradicting cycle.

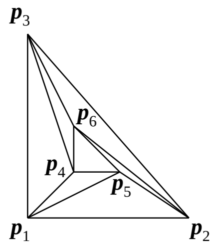


Fig. 2. Triangulation of Example 4.

Example 4 (Another nonregular triangulation with 6 vertices) The vertex $\boldsymbol{p}_2$ in Example 3 is perturbed. The point configuration becomes

$$\begin{aligned} \boldsymbol{p}_1 &= (0\ 0), & \boldsymbol{p}_2 &= (\tfrac{7}{2}\ 0), & \boldsymbol{p}_3 &= (0\ 4), \\ \boldsymbol{p}_4 &= (1\ 1), & \boldsymbol{p}_5 &= (2\ 1), & \boldsymbol{p}_6 &= (1\ 2). \end{aligned}$$

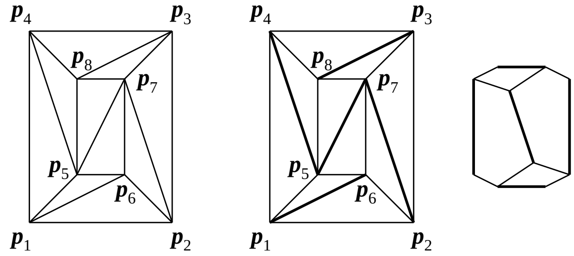
The triangulation Δ is indicated in Fig. 2 below. Each of the graph viewed from $\boldsymbol{v}_1 = (\frac{5}{4}\ \frac{3}{2})$, $\boldsymbol{v}_2 = (\frac{4}{3}\ \frac{4}{3})$, or $\boldsymbol{v}_3 = (\frac{7}{5}\ \frac{7}{5})$ has a unique contradicting cycle. The matrix representing the regularity of Δ is

$$A = \begin{array}{c|cccccc} & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ \hline \boldsymbol{p}_1\boldsymbol{p}_4 & 3 & & 1 & -8 & 4 & \\ \boldsymbol{p}_1\boldsymbol{p}_5 & -1 & 1 & & \frac{7}{2} & -\frac{7}{2} & \\ \boldsymbol{p}_2\boldsymbol{p}_5 & \frac{1}{2} & 3 & & & -7 & \frac{7}{2} \\ \boldsymbol{p}_2\boldsymbol{p}_6 & & -1 & \frac{1}{2} & & 3 & -\frac{5}{2} \\ \boldsymbol{p}_3\boldsymbol{p}_4 & 1 & & -1 & -4 & & 4 \\ \boldsymbol{p}_3\boldsymbol{p}_6 & & 1 & \frac{5}{2} & 3 & & -\frac{13}{2} \\ \boldsymbol{p}_4\boldsymbol{p}_5 & 1 & & & -3 & 1 & 1 \\ \boldsymbol{p}_4\boldsymbol{p}_6 & & & 1 & 1 & 1 & -3 \\ \boldsymbol{p}_5\boldsymbol{p}_6 & & 1 & & \frac{1}{2} & -\frac{5}{2} & 1 \end{array} .$$

The polyhedron of the solutions of the dual problem is a cone

$$\begin{aligned} &\{\boldsymbol{u} \geq \boldsymbol{0} : A\boldsymbol{u} = \boldsymbol{0}\} \\ &= \mathbb{R}_{\geq 0}(1\,8\,0\,8\,5\,0\,0\,0\,0) \\ &\quad + \mathbb{R}_{\geq 0}(0\,8\,2\,14\,7\,0\,0\,0\,0) \\ &\quad + \mathbb{R}_{\geq 0}(0\,6\,0\,7\,6\,1\,0\,0\,0) \\ &\quad + \mathbb{R}_{\geq 0}(0\,2\,0\,2\,1\,0\,1\,0\,0) \\ &\quad + \mathbb{R}_{\geq 0}(0\,2\,0\,2\,2\,0\,0\,1\,0) \\ &\quad + \mathbb{R}_{\geq 0}(0\,1\,0\,2\,1\,0\,0\,0\,1), \end{aligned}$$

where interior 1-simplices are indexed lexicographically. The first three rays correspond to the solutions made by the contradicting cycles in view graphs



(a) triangulation (b) support of the dual solutions

Fig. 3. Example 5.

$G_{\boldsymbol{v}_1}, G_{\boldsymbol{v}_2}, G_{\boldsymbol{v}_3}$, as in Subsection 2.2. The latter three rays have no such correspondence.

Example 5 (Counterexample for the reverse of the main theorem) With the point configuration

$$\boldsymbol{p}_1 = (0\,0), \quad \boldsymbol{p}_2 = (3\,0), \quad \boldsymbol{p}_3 = (3\,4), \quad \boldsymbol{p}_4 = (0\,4),$$
$$\boldsymbol{p}_5 = (1\,1), \quad \boldsymbol{p}_6 = (2\,1), \quad \boldsymbol{p}_7 = (2\,3), \quad \boldsymbol{p}_8 = (1\,3),$$

the triangulation Δ indicated in Fig. 3(a) below is a nonregular triangulation with none of its view graphs $G_{\boldsymbol{v}}$ containing a contradicting cycle. The matrix representing the regularity of Δ is

$A =$

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
$\boldsymbol{p}_1\boldsymbol{p}_5$	3			1	−8	4		
$\boldsymbol{p}_1\boldsymbol{p}_6$	−1	1			3	−3		
$\boldsymbol{p}_2\boldsymbol{p}_6$	2	4				−9	3	
$\boldsymbol{p}_2\boldsymbol{p}_7$		−2	2			4	−4	
$\boldsymbol{p}_3\boldsymbol{p}_7$		1	3				−8	4
$\boldsymbol{p}_3\boldsymbol{p}_8$			−1	1			3	−3
$\boldsymbol{p}_4\boldsymbol{p}_8$			2	4	3			−9
$\boldsymbol{p}_4\boldsymbol{p}_5$	2			−2	−4			4
$\boldsymbol{p}_5\boldsymbol{p}_6$	2				−4	1	1	
$\boldsymbol{p}_6\boldsymbol{p}_7$		2			2	−5	1	
$\boldsymbol{p}_7\boldsymbol{p}_8$			2		1		−4	1
$\boldsymbol{p}_5\boldsymbol{p}_8$				2	1		2	−5
$\boldsymbol{p}_5\boldsymbol{p}_7$					−2	2	−2	2

The polyhedron of the solutions of the dual problem is a single ray

$$\{\boldsymbol{u} \geq \boldsymbol{0} : A\boldsymbol{u} = \boldsymbol{0}\} = \mathbb{R}_{\geq 0}(0\,2\,0\,1\,0\,2\,0\,1\,0\,0\,0\,0\,1),$$

where interior 1-simplices are indexed lexicographically. The support of the nonzero solutions is denoted by bold edges in Fig. 3(b). If a contradicting cycle existed for some view graph $G_{\boldsymbol{v}}$, this (directed) cycle should contain all of the bold edges (in its underlying undirected counterpart). However, there are no cycles containing all of these bold edges. Hence, there exists no view graph $G_{\boldsymbol{v}}$ containing a contradicting cycle for this example. (Remark: If we take the edge $\boldsymbol{p}_6\boldsymbol{p}_8$ instead of $\boldsymbol{p}_5\boldsymbol{p}_7$, this new flipped triangulation becomes regular.)

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